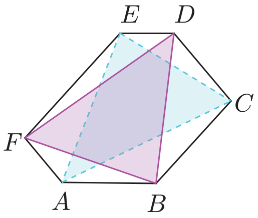
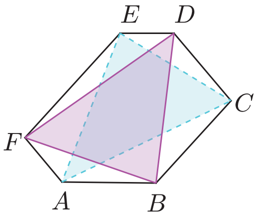
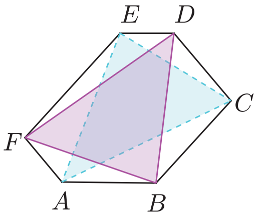
Příklad 12

Nechť $ABCDEF$ je konvexní šestiúhelník, jehož každé dvě protilehlé strany jsou rovnoběžné. Dokažte, že trojúhelníky ACE a BDF mají stejný obsah.





O – libovolný počátek,
$\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$, $\vec{d} = \overrightarrow{OD}$, $\vec{e} = \overrightarrow{OE}$, $\vec{f} = \overrightarrow{OF}$.



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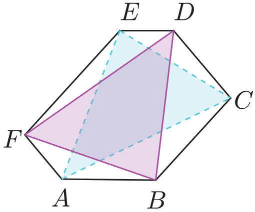
Z rovnoběžnosti protilehlých stran

$$\vec{o}=\overrightarrow{AB}\times\overrightarrow{ED}=(\vec{b}-\vec{a})\times(\vec{d}-\vec{e})=\vec{b}\times\vec{d}-\underline{\vec{b}\times\vec{e}}-\underline{\vec{a}\times\vec{d}}+\vec{a}\times\vec{e},$$



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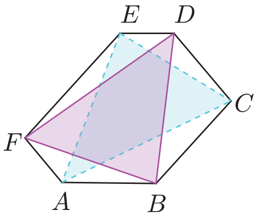
Z rovnoběžnosti protilehlých stran

$$\vec{o} = \overrightarrow{AB} \times \overrightarrow{ED} = (\vec{b} - \vec{a}) \times (\vec{d} - \vec{e}) = \vec{b} \times \vec{d} - \underline{\vec{b} \times \vec{e}} - \underline{\vec{a} \times \vec{d}} + \vec{a} \times \vec{e},$$

$$\vec{o} = \overrightarrow{CB} \times \overrightarrow{FE} = (\vec{b} - \vec{c}) \times (\vec{e} - \vec{f}) = \underline{\vec{b} \times \vec{e}} - \vec{b} \times \vec{f} - \vec{c} \times \vec{e} + \underline{\vec{c} \times \vec{f}},$$

$$\vec{o} = \overrightarrow{CD} \times \overrightarrow{AF} = (\vec{d} - \vec{c}) \times (\vec{f} - \vec{a}) = \vec{d} \times \vec{f} - \underline{\vec{d} \times \vec{a}} - \underline{\vec{c} \times \vec{f}} + \vec{c} \times \vec{a}.$$



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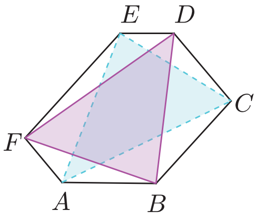
Z rovnoběžnosti protilehlých stran

$$\vec{o} = \overrightarrow{AB} \times \overrightarrow{ED} = (\vec{b} - \vec{a}) \times (\vec{d} - \vec{e}) = \vec{b} \times \vec{d} - \underline{\vec{b} \times \vec{e}} - \underline{\vec{a} \times \vec{d}} + \vec{a} \times \vec{e},$$

$$\vec{o} = \overrightarrow{CB} \times \overrightarrow{FE} = (\vec{b} - \vec{c}) \times (\vec{e} - \vec{f}) = \underline{\vec{b} \times \vec{e}} - \vec{b} \times \vec{f} - \vec{c} \times \vec{e} + \underline{\vec{c} \times \vec{f}},$$

$$\vec{o} = \overrightarrow{CD} \times \overrightarrow{AF} = (\vec{d} - \vec{c}) \times (\vec{f} - \vec{a}) = \vec{d} \times \vec{f} - \underline{\vec{d} \times \vec{a}} - \underline{\vec{c} \times \vec{f}} + \vec{c} \times \vec{a}.$$

Sečtením a úpravou: $\quad \vec{c} \times \vec{e} - \vec{c} \times \vec{a} - \vec{a} \times \vec{e} = \vec{d} \times \vec{f} - \vec{d} \times \vec{b} - \vec{b} \times \vec{f}.$



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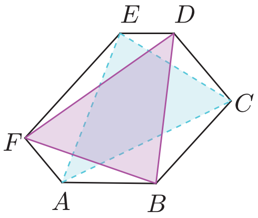
$$\boxed{\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e} = \vec{d}\times\vec{f}-\vec{d}\times\vec{b}-\vec{b}\times\vec{f}}$$



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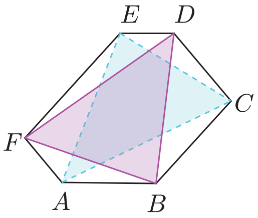
$$\boxed{\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e}=\vec{d}\times\vec{f}-\vec{d}\times\vec{b}-\vec{b}\times\vec{f}}$$

$$2S_{ACE}=\left|\overrightarrow{AC}\times\overrightarrow{AE}\right|=|(\vec{c}-\vec{a})\times(\vec{e}-\vec{a})|=|\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e}|,$$



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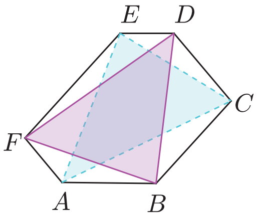
$$\boxed{\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e} = \vec{d}\times\vec{f}-\vec{d}\times\vec{b}-\vec{b}\times\vec{f}}$$

$$2S_{ACE} = \left|\overrightarrow{AC}\times\overrightarrow{AE}\right| = |(\vec{c}-\vec{a})\times(\vec{e}-\vec{a})| = |\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e}|,$$

$$2S_{BDF} = \left|\overrightarrow{BD}\times\overrightarrow{BF}\right| = |(\vec{d}-\vec{b})\times(\vec{f}-\vec{b})| = |\vec{d}\times\vec{f}-\vec{d}\times\vec{b}-\vec{b}\times\vec{f}|\,.$$



O – libovolný počátek,
$\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$, $\vec{d} = \overrightarrow{OD}$, $\vec{e} = \overrightarrow{OE}$, $\vec{f} = \overrightarrow{OF}$.

$$\boxed{\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e} = \vec{d}\times\vec{f}-\vec{d}\times\vec{b}-\vec{b}\times\vec{f}}$$

$$2S_{ACE} = \left|\overrightarrow{AC}\times\overrightarrow{AE}\right| = |(\vec{c}-\vec{a})\times(\vec{e}-\vec{a})| = |\vec{c}\times\vec{e}-\vec{c}\times\vec{a}-\vec{a}\times\vec{e}|,$$

$$2S_{BDF} = \left|\overrightarrow{BD}\times\overrightarrow{BF}\right| = |(\vec{d}-\vec{b})\times(\vec{f}-\vec{b})| = |\vec{d}\times\vec{f}-\vec{d}\times\vec{b}-\vec{b}\times\vec{f}|\,.$$

Proto skutečně $\quad S_{ACE} = S_{BDF}. \quad \square$