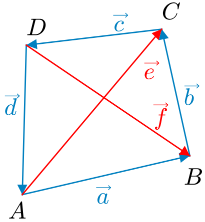
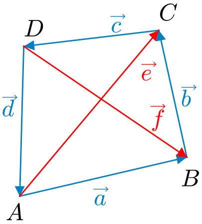
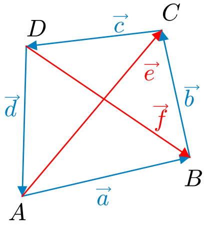
Příklad 5

Úhlopříčky čtyřúhelníku $ABCD$ jsou navzájem kolmé, právě když pro délky jeho stran platí rovnost $a^2 + c^2 = b^2 + d^2$. Dokažte.



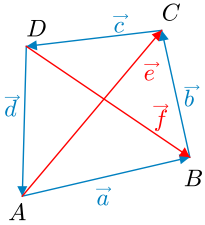


$$a^2 + c^2 = b^2 + d^2 \quad \Leftrightarrow \quad |\vec{a}|^2 + |\vec{c}|^2 - \left|\vec{b}\right|^2 - \left|\vec{d}\right|^2 = 0 \quad \Leftrightarrow$$



$$a^2 + c^2 = b^2 + d^2 \quad \Leftrightarrow \quad |\vec{a}|^2 + |\vec{c}|^2 - \left|\vec{b}\right|^2 - \left|\vec{d}\right|^2 = 0 \quad \Leftrightarrow$$

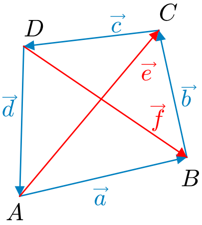
$$\Leftrightarrow \quad \left|\vec{e} - \vec{b}\right|^2 + \left|-\vec{e} - \vec{d}\right|^2 - \left|\vec{b}\right|^2 - \left|\vec{d}\right|^2 = 0 \quad \Leftrightarrow$$



$$a^2+c^2=b^2+d^2 \quad \Leftrightarrow \quad |\vec{a}|^2+|\vec{c}|^2-\left|\vec{b}\right|^2-\left|\vec{d}\right|^2=0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left|\vec{e}-\vec{b}\right|^2+\left|-\vec{e}-\vec{d}\right|^2-\left|\vec{b}\right|^2-\left|\vec{d}\right|^2=0 \quad \Leftrightarrow$$

$$\Leftrightarrow \; |\vec{e}|^2+\underline{\left|\vec{b}\right|}^2-2\vec{e}\cdot\vec{b}+|\vec{e}|^2+\underline{\left|\vec{d}\right|}^2+2\vec{e}\cdot\vec{d}-\underline{\left|\vec{b}\right|}^2-\underline{\left|\vec{d}\right|}^2=0 \; \Leftrightarrow$$



$$a^2 + c^2 = b^2 + d^2 \quad \Leftrightarrow \quad |\vec{a}|^2 + |\vec{c}|^2 - \left|\vec{b}\right|^2 - \left|\vec{d}\right|^2 = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left|\vec{e} - \vec{b}\right|^2 + \left|-\vec{e} - \vec{d}\right|^2 - \left|\vec{b}\right|^2 - \left|\vec{d}\right|^2 = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \; |\vec{e}|^2 + \underline{\left|\vec{b}\right|^2} - 2\vec{e}\cdot\vec{b} + |\vec{e}|^2 + \underline{\left|\vec{d}\right|^2} + 2\vec{e}\cdot\vec{d} - \underline{\left|\vec{b}\right|^2} - \underline{\left|\vec{d}\right|^2} = 0 \; \Leftrightarrow$$

$$\Leftrightarrow \; 2\vec{e}\cdot\left(\vec{e} - \vec{b} + \vec{d}\,\right) = 0 \; \Leftrightarrow \; \vec{e}\cdot\left(-\vec{c} - \vec{b}\right) = 0 \; \Leftrightarrow \; \vec{e}\cdot\vec{f} = 0 \quad \square$$